The second approximation to cnoidal and solitary waves

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The expansion method introduced by Friedrichs (1948) for the systematic development of shallow-water theory for water waves of large wavelength was used by Keller (1948) to obtain the first approximation for the finite-amplitude solitary wave of Boussinesq (1872) and Rayleigh (1876), as well as for periodic waves of permanent type, corresponding to the cnoidal waves of Korteweg & de Vries (1895).

The present investigation extends Friedrich's method so as to include terms up to the fourth order from shallow-water theory for a flat horizontal bottom, and thereby obtains the complete second approximations to both cnoidal and solitary waves. These second approximations show that, unlike the first approximation, the vertical motions cannot be considered as negligible, and that the pressure variation is no longer hydrostatic.

1. Introduction

This paper is primarily concerned with higher-order solutions of the finite-amplitude, long water waves which are propagated without a change in shape in shallow water. We will obtain the exact second approximations to the solitary wave first analysed by Boussinesq (1872) and Rayleigh (1876), and to the periodic waves of permanent form, the so-called 'cnoidal' waves, which were discovered by Korteweg & de Vries (1895).

The successive approximations are obtained through Friedrichs's (1948) expansion method for shallow-water theory. This method consists of a power series development in terms of a dimensionless parameter which is related to the transformations that are found necessary in order to yield the classical non-linear shallow-water equations from the zero-order terms in this small perturbation series. Friedrichs's expansion method was analysed and discussed by Stoker (1957), and was used by Keller (1948) to obtain the first approximation to the solitary wave and periodic waves resembling the cnoidal waves. It will be shown that if only the first-order terms are retained, then the solution obtained by Friedrichs's (1948) method is identical to the one first given by Korteweg & de Vries (1895).

The second approximations to the solitary and cnoidal waves will be shown to depend on some of the fourth-order terms in the Friedrichs's expansion method. These second approximations will show that, in the higher-order terms, the variation in pressure is no longer hydrostatic, and that the vertical velocity and vertical accelerations are no longer small in magnitude. These relations are used to indicate that probably the maximum elevation of any wave of finite amplitude is $\frac{8}{11}$ of the depth below the trough. This indicates that the limiting total height of a solitary wave may be $1{\cdot}727h$, where h is the undisturbed water depth. This value is in much better agreement with the experimental data of Daily & Stephan (1952) than is the commonly used value of 1·782, as obtained by McCowan (1894), or the value of 1·827 as recently derived by Yamada (1957).

The use of Friedrichs (1948) expansion method for these particular shallow-water investigations has been shown to have at least an asymptotic significance by the existence proof of Friedrichs & Hyers (1954) for solitary waves, and the corresponding existence proof by Littman (1957) for cnoidal waves. Truesdell (1956, p. xcviii) has also pointed out that Friedrichs's expansion method for shallow-water waves is only a particular case of a more general perturbation series developed by Lagrange.

2. Friedrichs's expansion method for non-linear shallow-water theory

If we follow the procedure introduced by Friedrichs (1948), we 'stretch' the vertical independent variable y with respect to the horizontal independent variable x by introducing the following non-dimensional variables based upon reference lengths d and e with $d \ll e$:

$$\left.\begin{aligned} &X = \frac{x}{e}, \quad Y = \frac{y}{d}, \\ &U = \frac{u}{\sqrt{(gd)}}, \quad V = \frac{v}{\sqrt{(gd)}}\left(\frac{d}{e}\right), \\ &H = \frac{h}{d}, \quad N = \frac{\eta}{d}, \quad P = \frac{\Delta p}{\rho g d}. \end{aligned}\right\} \tag{2.1}$$

These non-dimensional variables are then introduced into the continuity equation, the equations of motion, the potential or irrotationality condition, and the boundary conditions at the free surface $y = \eta$ and at the flat horizontal rigid bottom $y = -h = \text{const.}$ (see figure 1), which are written as follows in terms of the Euler variables for steady flow in the direction of the x-axis only (e.g. see Stoker (1957, p. 28) or Lamb (1932)):

$$\left.\begin{aligned} u_x + v_y &= 0, \\ uu_x + vu_y &= -\frac{1}{\rho}p_x, \\ uv_x + vv_y &= -\frac{1}{\rho}p_y - g, \\ v_x &= u_y, \\ \Delta p(x,\eta) &= p(x,\eta) - p_{\text{atmos.}} = 0, \\ v(x,\eta) &= u(x,\eta)\,\eta_x, \quad v(x,-h) = 0. \end{aligned}\right\} \tag{2.2}$$

In terms of the non-dimensional stretching parameter defined by (2.1) and

$$\sigma = (d/e)^2 \ll 1, \tag{2.3}$$

equations (2.2) are transformed to

$$\left.\begin{aligned} \sigma U_X + V_Y &= 0, \\ \sigma(UU_X + P_X) + VU_Y &= 0, \\ \sigma(UV_X + P_Y + 1) + VV_Y &= 0, \\ V_X = U_Y, \quad P(X,N) &= 0, \\ V(X,N) = \sigma U(X,N)\,N_X, \quad V(X,-H) &= 0. \end{aligned}\right\} \tag{2.4}$$

Now, following Friedrichs (1948), we assume that U, V, and P each has a power series expansion in terms of the stretching parameter defined by (2.3) in the form

$$\left.\begin{aligned} F(X,Y) &= \sum_{n=0}^{\infty} \sigma^n F_n(X,Y) \\ \text{for} \qquad -H \leqslant Y \leqslant N(X) &= \sum_{r=0}^{\infty} \sigma^r N_r(X). \end{aligned}\right\} \tag{2.5}$$

Since H is known (a constant for the flat horizontal bottom case being considered), therefore U and V can be directly evaluated at $Y = -H$. However, at the free surface $Y = N(X)$, we cannot directly evaluate U, V, or P since N is unknown and changing directly with each N_r for every order of approximation. Consequently, we must follow the procedure of Friedrichs (1948) and Keller (1948) by evaluating U, V, P and all their derivatives at the zero-order elevation given by $N_0(X)$. Now we will generalize this procedure, so as to directly obtain the terms of a higher order than given by Keller (1948), by expressing either U, V, or P at the surface by means of the Taylor series expansion of each term of (2.5). Thus,

$$\begin{aligned} F(X,N) &= \sum_{n=0}^{\infty} \sigma^n F_n(X,N) \\ &= \sum_{n=0}^{\infty} \sigma^n \sum_{m=0}^{\infty} \frac{(N-N_0)^m}{m!}\left[\frac{\partial^m F_n(X,Y)}{\partial Y^m}\right]_{Y=N_0}, \end{aligned}$$

with

$$N - N_0 = \sum_{r=1}^{\infty} \sigma^r N_r(X). \tag{2.6}$$

Therefore, to the fourth order, we have for either U, V, or P

$$\begin{aligned} F(X,N) = [&F_0 + \sigma(N_1 F_{0_Y} + F_1) + \sigma^2(N_2 F_{0_Y} + \tfrac{1}{2}N_1^2 F_{0_{YY}} + N_1 F_{1_Y} + F_2) \\ &+ \sigma^3(N_3 F_{0_Y} + N_1 N_2 F_{0_{YY}} + \tfrac{1}{6}N_1^3 F_{0_{YYY}} + N_2 F_{1_Y} + \tfrac{1}{2}N_1^2 F_{1_{YY}} + N_1 F_{2_Y} + F_3) \\ &+ \sigma^4(N_4 F_{0_Y} + \tfrac{1}{2}N_2^2 F_{0_{YY}} + N_1 N_3 F_{0_{YY}} + \tfrac{1}{2}N_1^2 N_2 F_{0_{YYY}} + \tfrac{1}{24}N_1^4 F_{0_{YYYY}} \\ &+ N_3 F_{1_Y} + N_1 N_2 F_{1_{YY}} + \tfrac{1}{6}N_1^3 F_{1_{YYY}} + N_2 F_{2_Y} + \tfrac{1}{2}N_1^2 F_{2_{YY}} + N_1 F_{3_Y} + F_4)]_{Y=N_0}. \end{aligned} \tag{2.7}$$

Then, upon introducing (2.5) and (2.7) into (2.4), we obtain for the zero-order terms

$$\left.\begin{aligned} V_{0_Y} = 0, \quad V_0 U_{0_Y} &= 0 = V_0 V_{0_Y}, \\ V_{0_X} = U_{0_Y}, \quad P_0(X,N_0) &= 0, \\ V_0(X,N_0) = 0, \quad V_0(X,-H) &= 0; \end{aligned}\right\} \tag{2.8}$$

which may be solved to yield the following:

$$U_0 = U_0(X), \quad V_0(X,Y) = 0, \quad P_0(X,N_0) = 0. \tag{2.9}$$

With these relations kept in mind, the σ terms from (2.4) reduce to

$$\left.\begin{aligned} U_{0_X}+V_{1_Y} &= 0, \\ U_0U_{0_X}+P_{0_X} = 0, \quad P_{0_Y} &= -1, \\ V_{1_X} &= U_{1_Y}, \\ P_1(X,N_0)+N_1P_{0_Y} &= 0, \\ V_1(X,N_0) = U_0N_{0_X}, \quad V_1(X,-H) &= 0. \end{aligned}\right\} \tag{2.10}$$

Now if we restrict ourselves to the steady-state finite-amplitude waves that are defined by $U_0 = \text{const.}$, we have from (2.9) and (2.10)

$$\left.\begin{aligned} U_0 = \text{const.}, \quad V_0 = 0 = V_1, \quad U_1 = f(X), \\ N_0 = \text{const.}, \quad P_0(X,Y) = N_0 - Y, \quad P_1(X,N_0) = N_1(X). \end{aligned}\right\} \tag{2.11}$$

As shown by Keller (1948) this is the only finite-amplitude solution to the first order of approximation. We can duplicate his results by obtaining the σ^2 terms of (2.4) as

$$\left.\begin{aligned} U_{1_X}+V_{2_Y} &= 0, \\ U_0U_{1_X}+P_{1_X} = 0, \quad P_{1_Y} &= 0, \\ V_{2_X} = U_{2_Y}, \quad P_2(X,N_0)-N_2 &= 0, \\ V_2(X,N_0) = U_0N_{1_X}, \quad V_2(X,-H) &= 0, \end{aligned}\right\} \tag{2.12}$$

which may be integrated to yield

$$-P_1(X) = \int U_0U_{1_X}dX = U_0f(X)+C = -N_1(X), \tag{2.13}$$

$$V_2(X,Y) = -\int_{-H}^{Y} U_{1_X}dY = -(Y+H)f_X, \tag{2.14}$$

$$-N_1(X) = \int -\frac{V_2(X,N_0)}{U_0}dX = \frac{N_0+H}{U_0}f(X)+C. \tag{2.15}$$

Consequently, the identities for N_1 in (2.13) and (2.15) show that U_0 is restricted to the unique constant value given by

$$U_0 = \surd(N_0+H), \quad u_0 = \surd[g(\eta_0+h)]. \tag{2.16}$$

This proves that the only non-trivial first-order finite-amplitude solution would correspond to a hydraulic jump having $N_{0_X} = 0$ on both sides of the discontinuity.

The second-order solution can now be found by evaluating $f(X)$ in the same manner as just used to find U_0. That is, we find one expression for N_2 from the free surface boundary condition defined by constant pressure on the surface, or $P_2(X,N) = 0$, and another expression for N_2 from the corresponding free surface boundary condition for $V_3(X,N)$. Then, by equating these two identities, we will find a differential equation to be satisfied by $f(X)$.

3. The first approximation to the cnoidal and solitary waves

In somewhat the same manner as just outlined in § 2, Keller (1948) used Friedrichs's (1948) expansion to obtain the first approximation to finite amplitude waves similar to the periodic cnoidal waves of Korteweg & de Vries (1895), and similar to the solitary wave of Boussinesq (1872) and Rayleigh (1876). It will now be shown that if the proper order of terms are retained throughout, then Friedrichs's expansion method, as introduced in § 2, will give exactly the same first approximation to finite-amplitude waves as originally given by Korteweg & de Vries.

The σ^3 terms of (2.4), as obtained by introducing (2.5), (2.7) and (2.11), are now found to be the following:

$$\left.\begin{aligned} U_{2_X}+V_{3_Y} &= 0,\\ U_0U_{2_X}+U_1U_{1_X}+P_{2_X} &= 0,\\ U_0V_{2_X}+P_{2_Y} &= 0,\\ V_{3_X} &= U_{3_Y},\\ P_3(X,N_0)-N_3+N_1P_{2_Y} &= 0,\\ V_3(X,N_0) = U_0N_{2_X}+U_1N_{1_X}-N_1V_{2_Y} &= U_0N_{2_X}-[(U_0f+C)f]_X,\\ V_3(X,-H) &= 0. \end{aligned}\right\} \tag{3.1}$$

The second and third equations in (3.1) may be integrated to yield

$$P_2(X,Y) = -U_0U_2-\tfrac{1}{2}f^2+\text{const.} \tag{3.2}$$

Then, upon substituting (3.2) into (2.12), we obtain

$$N_2(X) = P_2(X,N_0) = [\tfrac{1}{2}U_0(N_0^2+2HN_0)f_{XX}-U_0R-\tfrac{1}{2}f^2+\text{const.}], \tag{3.3}$$

since we find from (2.12) and (2.14) that

$$U_2(X,Y) = \int V_{2_X}dY = -\int (Y+H)f_{XX}dY = -\tfrac{1}{2}(Y^2+2HY)f_{XX}+R(X). \tag{3.4}$$

Another expression for N_2 may now be derived by integrating the first equation in (3.1) with $V_3(X,-H)=0$ to obtain

$$V_3(X,Y) = -\int_{-H}^{Y} U_{2_X}dY = [\tfrac{1}{6}(Y^3+3HY^2-2H^3)f_{XXX}-(Y+H)R_X]. \tag{3.5}$$

Then, by substituting this into the boundary condition for $V_3(X,N_0)$ in (3.1), we finally obtain, after integration with respect to X,

$$N_2(X) = [U_0f^2+Cf+\tfrac{1}{6}(N_0^3+3HN_0^2-2H^3)f_{XX}-(N_0+H)R](U_0)^{-1}+\text{const.} \tag{3.6}$$

We then combine (3.3) and (3.6) to find the ordinary differential equation defining $f(X)$:

$$f_{XX}-\left(\frac{9}{2U_0^5}\right)f^2-\left(\frac{3C}{U_0^6}\right)f+C_0 = 0, \tag{3.7}$$

after noticing that $R(X)$ is eliminated by introducing (2.16).

The periodic solution of (3.7) for cnoidal waves is given by the square of the Jacobian elliptic function cn, which has the modulus k and the real period defined by $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. This solution is

$$\left.\begin{aligned} f(X) &= -\tfrac{4}{3}U_0^5\alpha^2k^2cn^2(\alpha X,k),\\ C &= \tfrac{4}{3}U_0^6\alpha^2(2k^2-1),\\ C_0 &= \tfrac{8}{3}U_0^5\alpha^4k^2(1-k^2) \geqslant 0. \end{aligned}\right\} \tag{3.8}$$

The limiting case of $k = 1$, $C_0 = 0$ corresponds to the solitary wave solution of Boussinesq (1872) and Rayleigh (1876) since

$$cn(\alpha X, 1) = \operatorname{sech} \alpha X. \tag{3.9}$$

However, it is obvious that the other limit of $k = 0$, namely

$$cn(\alpha X, 0) = \cos \alpha X, \tag{3.10}$$

cannot provide an explicit solution of (3.7). This is in agreement with the fact that the trigonometric functions cannot form an exact, explicit solution for finite-amplitude, steady-state waveforms, as first proved by Korteweg & de Vries (1895).

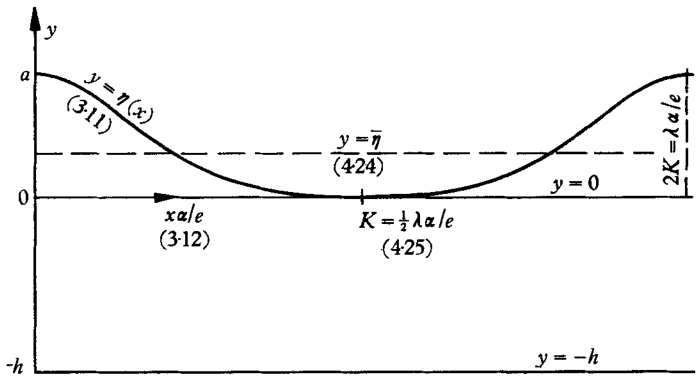

FIGURE 1. The cnoidal wave for

$k^2 = 0{\cdot}9$, $a/h = 0{\cdot}7129$, $2K/\lambda = \alpha/e = 0{\cdot}771/h$, $\lambda/h = 6{\cdot}68 < 10$, $\bar{\eta}/h = 0{\cdot}260$, $\eta/h = 0{\cdot}7129\, cn^2(0{\cdot}771x/h)$, $C/\surd(gh) = 1{\cdot}057$.

The constants C and N_0 can now be evaluated from the boundary conditions for the cnoidal wave, as shown in figure 1, by combining (2.5), (2.15) and (3.8) as

$$\begin{aligned}\frac{\eta(X)}{h} &= N(X)\frac{d}{h} = [N_0 + \sigma N_1 + O(\sigma^2)]\frac{d}{h} \\ &= \frac{\eta_0}{h} + \frac{d^3}{e^2 h}(\tfrac{4}{3}U_0^6\alpha^2 k^2)\left[cn^2(\alpha X, k) - \left(\frac{2k^2-1}{k^2}\right)\right] \\ &= \frac{a}{h}cn^2(\alpha X, k) + O\left(\frac{a}{h}\right)^2,\end{aligned} \tag{3.11}$$

where
$$\alpha X = \frac{\alpha x}{e} = x\left[\frac{3}{4k^2}\frac{a}{(dU_0^2)^3}\right]^{\frac{1}{2}} = \frac{x}{h}\left[\frac{3}{4k^2}\left(\frac{a}{h}\right)\right]^{\frac{1}{2}} + O\left(\frac{a}{h}\right)^{\frac{3}{2}}, \tag{3.12}$$

since
$$\frac{dU_0^2}{h} = 1 + \frac{\eta_0}{h},$$

$$\frac{\eta_0}{h} = \left(\frac{2k^2-1}{k^2}\right)\frac{a}{h} = \left(\frac{2k^2-1}{k^2}\right)\frac{d^3}{e^2 h}(\tfrac{4}{3}U_0^6\alpha^2 k^2). \tag{3.13}$$

The complete solution of the first approximation to cnoidal waves may therefore be obtained as

$$\frac{\Delta p(x,y)}{\rho g h} = [P_0 + \sigma P_1 + O(\sigma^2)]\frac{d}{h} = \frac{\eta(x) - y}{h} + O\left(\frac{a}{h}\right)^2, \tag{3.14}$$

$$\frac{u(x)}{\sqrt{(gh)}} = [U_0 + \sigma U_1 + O(\sigma^2)]\left(\frac{d}{h}\right)^{\frac{1}{2}} = 1 + \left(\frac{2k^2-1}{2k^2}\right)\frac{a}{h} - \frac{\eta(x)}{h} + O\left(\frac{a}{h}\right)^2, \tag{3.15}$$

$$\begin{aligned}\frac{v(x,y)}{\sqrt{(gh)}} &= [\sigma^2 V_2 + O(\sigma^3)]\left(\frac{d}{\sigma h}\right)^{\frac{1}{2}} = \left(1+\frac{y}{h}\right)\frac{d\eta(x)}{dx} + O\left(\frac{a}{h}\right)^{\frac{5}{2}} \\ &= -\left(1+\frac{y}{h}\right)\left[\frac{3}{k^2}\left(\frac{a}{h}\right)^3\right]^{\frac{1}{2}} cn(\alpha X, k)\, sn(\alpha X, k)\, dn(\alpha X, k) + O\left(\frac{a}{h}\right)^{\frac{5}{2}}. \end{aligned}\tag{3.16}$$

This first approximation to the cnoidal wave (or the solitary wave for $k = 1$) is identical to the solutions originally given by Korteweg & de Vries (1895) if one correctly eliminates their higher-order terms which will be shown to be incorrect. It is interesting to note that the pressure is still hydrostatic but the vertical velocity cannot be neglected, as commonly assumed. Although the vertical velocity magnitude is of a slightly higher order of approximation, it is not a negligible quantity of order $(a/h)^2$. We were able to evaluate the vertical velocity variation, which is seen to be a linear function of the water depth, since V_2 does not depend upon the unknown function $R(X)$ which we must now evaluate in order to determine the next approximation to the cnoidal wave.

4. The second approximation to the cnoidal and solitary waves

In order to obtain the next, or second approximation we now need the σ^4 terms of (2.4), which may be obtained by the same procedure as used in § 3. Thus,

$$\left.\begin{aligned} U_{3_X} + V_{4_Y} &= 0, \\ U_0 U_{3_X} + U_1 U_{2_X} + U_2 U_{1_X} + P_{3_X} + V_2 U_{2_Y} &= 0, \\ U_0 V_{3_X} + U_1 V_{2_X} + P_{3_Y} + V_2 V_{2_Y} &= 0, \\ U_{4_Y} &= V_{4_X}, \\ P_4(X, N_0) - N_4 + N_2 P_{2_Y} + \tfrac{1}{2} N_1^2 P_{2_{YY}} + N_1 P_{3_Y} &= 0, \\ V_4(X, N_0) = U_0 N_{3_X} + U_1 N_{2_X} + U_2 N_{1_X} &- N_2 V_{2_Y} - N_1 V_{3_Y} \\ = [U_0 N_{3_X} + (N_2 U_1)_X &+ (N_1 U_2)_X]_{Y=N_0}, \\ V_4(X, -H) &= 0. \end{aligned}\right\} \tag{4.1}$$

The second and third equations in (4.1) may be integrated to yield

$$P_3(X, Y) = -U_0 U_3 - U_1 U_2 - \tfrac{1}{2} V_2^2 + \text{const.}, \tag{4.2}$$

so that the fifth equation in (3.1) may then be written as

$$\begin{aligned} N_3(X) &= P_3(X, N_0) + N_1 P_{2_Y} \\ &= \Big\{ -U_0 S(X) - \frac{U_0}{24}(N_0^4 + 4HN_0^3 - 8H^3 N_0) f_{XXXX} + \frac{U_0}{2}(N_0^2 + 2HN_0) R_{XX} \\ &\quad + \tfrac{1}{2}(N_0^2 + 2HN_0) f f_{XX} - fR - \frac{U_0^4}{2} f_x^2 - (U_0 f + C)\, U_0^3 f_{XX} + \text{const.}\Big\}, \end{aligned}\tag{4.3}$$

after noting that U_3 may be obtained from (3.1) and (3.5) as

$$U_3(X, Y) = \int V_{3_X} dY$$
$$= \{\tfrac{1}{24}(Y^4 + 4HY^3 - 8H^3Y) f_{XXXX} - \tfrac{1}{2}(Y^2 + 2HY) R_{XX} + S(X)\}. \quad (4.4)$$

Another expression for N_3 is developed from the first and last two equations in (4.1) by writing

$$V_4(X, Y) = -\int_{-H}^{Y} U_{3_X} dY$$
$$= \{-\tfrac{1}{120}(Y^5 + 5HY^4 - 20H^3Y^2 + 16H^5) f_{XXXXX} + \tfrac{1}{6}(Y^3 + 3HY^2 - 2H^3) R_{XXX} - (Y+H) S_X\}, \quad (4.5)$$

$$N_3(X) = \left\{\left[\int V_4(X, N_0)\, dX - N_2 U_1 - N_1 U_2\right] U_0^{-1}\right\}$$
$$= \left\{-\frac{1}{120U_0}(N_0^5 + 5HN_0^4 - 20H^3N_0^2 + 16H^5) f_{XXXX} + \frac{1}{6U_0}(N_0^3 + 3HN_0^2 - 2H^3) R_{XX} - U_0 S(X) - \frac{1}{U_0} N_2 f + (U_0 f + C)[-\tfrac{1}{2}(N_0^2 + 2HN_0) f_{XX} + R]\frac{1}{U_0} + \text{const.}\right\}. \quad (4.6)$$

Then, by taking N_2 from (3.3), we may equate (4.3) and (4.6) to obtain the following ordinary differential equation to be satisfied by $R(X)$:

$$\frac{U_0^5}{3} R_{XX} - \left(\frac{C}{U_0} + 3f\right) R = \frac{U_0^5}{30}(U_0^4 - 5H^2) f_{XXXX} - \tfrac{1}{2}(U_0^4 - 3H^2) f f_{XX} + \frac{C}{2U_0}(U_0^4 + H^2) f_{XX} + \frac{U_0^4}{2} f_x^2 + \frac{1}{2}\frac{f^3}{U_0} + \text{const.} \quad (4.7)$$

The solution of (4.7) that corresponds to (3.8) is given by

$$R(X) = \frac{C^2}{U_0^3}\left\{\left(\frac{k^2}{2k^2-1}\right)^2 \left(1 - \frac{9}{4}\frac{H^2}{U_0^4}\right) cn^4(\alpha X, k) + \left(\frac{k^2}{2k^2-1}\right)\left(1 + \frac{3}{2}\frac{H^2}{U_0^4}\right) cn^2(\alpha X, k) - \frac{3}{10}\frac{k^2(1-k^2)}{(2k^2-1)^2}\left(1 - \frac{5}{2}\frac{H^2}{U_0^4}\right) - \frac{3}{5}\right\}, \quad (4.8)$$

so (3.3), or (3.6), becomes

$$N_2(X) = \left(\frac{C}{U_0}\right)^2 \left\{\frac{3}{4}\left(\frac{k^2}{2k^2-1}\right)^2 cn^4(\alpha X, k) - \frac{5}{2}\left(\frac{k^2}{2k^2-1}\right) cn^2(\alpha X, k) + \frac{12 - 57k^2 + 57k^4}{20(2k^2-1)^2}\right\}, \quad (4.9)$$

which we now may add to (3.11) to give the second approximation as

$$\frac{\eta(X)}{h} = \frac{\eta_0}{h} + \frac{d^3}{e^2h} N_1 + \frac{d^5}{e^4h} N_2 + O\left(\frac{d^7}{e^6h}\right)$$
$$= \left\{\frac{\eta_0}{h} - \frac{\eta_1}{h}\left[1 - \left(\frac{k^2}{2k^2-1}\right) cn^2(\alpha X, k)\right] + \frac{\eta_1^2}{h(\eta_0+h)}\left[\frac{3}{4}\left(\frac{k^2}{2k^2-1}\right)^2 cn^4(\alpha X, k) - \frac{5}{2}\left(\frac{k^2}{2k^2-1}\right) cn^2(\alpha X, k) + \frac{12 - 57k^2 + 57k^4}{20(2k^2-1)^2}\right]\right\}, \quad (4.10)$$

where $$\eta_1 = \frac{Cd^3}{e^2} = \left(\frac{\alpha}{e}\right)^2 \tfrac{4}{3}(2k^2-1)(dU_0^2)^3 = \left(\frac{\alpha}{e}\right)^2 \tfrac{4}{3}(2k^2-1)(\eta_0+h)^3. \quad (4.11)$$

The periodic boundary conditions shown in figure 1 now yield the relations

$$\eta(0) = a = \left\{\eta_0 - \eta_1\left(\frac{k^2-1}{2k^2-1}\right) + \frac{\eta_1^2}{\eta_0+h}\left[\frac{12-7k^2-28k^4}{20(2k^2-1)^2}\right]\right\}, \tag{4.12}$$

$$\eta\left(\frac{K}{\alpha}\right) = O = \left\{\eta_0 - \eta_1 + \frac{\eta_1^2}{\eta_0+h}\left[\frac{12-57k^2+57k^4}{20(2k^2-1)^2}\right]\right\}, \tag{4.13}$$

which may be solved for the second approximation as

$$\frac{\eta_0}{h} = \frac{2k^2-1}{k^2}\frac{a}{h} + \left(\frac{a}{h}\right)^2\left(\frac{38-128k^2+113k^4}{20k^4}\right) + O\left(\frac{a}{h}\right)^3, \tag{4.14}$$

$$\frac{\eta_1}{h} = \frac{2k^2-1}{k^2}\frac{a}{h}\left[1+\frac{a}{h}\left(\frac{85k^2-50}{20k^2}\right)\right] + O\left(\frac{a}{h}\right)^3, \tag{4.15}$$

$$\frac{\eta(X)}{h} = \frac{a}{h}cn^2(\alpha X,k) - \frac{3}{4}\left(\frac{a}{h}\right)^2 cn^2(\alpha X,k)\left[1-cn^2(\alpha X,k)\right] + O\left(\frac{a}{h}\right)^3, \tag{4.16}$$

$$\begin{aligned}\alpha X &= \frac{x}{h}\left(\frac{3}{4k^2}\frac{a}{h}\right)^{\frac{1}{2}}\left(1+\frac{\eta_0}{h}\right)^{-\frac{3}{2}}\left[1+\frac{a}{h}\left(\frac{85k^2-50}{40k^2}\right)\right]\\ &= \frac{x}{h}\left(\frac{3}{4k^2}\frac{a}{h}\right)^{\frac{1}{2}}\left[1-\frac{a}{h}\left(\frac{7k^2-2}{8k^2}\right)\right] + O\left(\frac{a}{h}\right)^{\frac{5}{2}},\end{aligned} \tag{4.17}$$

$$\begin{aligned}\frac{u(x,y)}{\sqrt{(gh)}} &= \left\{1+\frac{d^2}{e^2U_0}f+\frac{d^4}{e^4U_0}\left[R-\frac{h^2}{2d^2}\left(2\frac{y}{h}+\frac{y^2}{h^2}\right)f_{XX}\right]\right\}\left(1+\frac{\eta_0}{h}\right)^{\frac{1}{2}}\\ &= \left\{1+\left(1-\frac{1}{2k^2}\right)\frac{a}{h}-\left(\frac{21k^4-6k^2-9}{40k^4}\right)\left(\frac{a}{h}\right)^2\right.\\ &\quad -\frac{a}{h}\left[1-\frac{a}{h}\left(\frac{7k^2-2}{4k^2}\right)-\frac{3}{2}\frac{a}{h}\left(2-\frac{1}{k^2}\right)\left(2\frac{y}{h}+\frac{y^2}{h^2}\right)\right]cn^2(\alpha X,k)\\ &\quad -\left(\frac{a}{h}\right)^2\left[\frac{5}{4}+\frac{9}{4}\left(2\frac{y}{h}+\frac{y^2}{h^2}\right)\right]cn^4(\alpha X,k)\\ &\quad \left.+\frac{3}{4}\left(\frac{a}{h}\right)^2\left(\frac{1}{k^2}-1\right)\left(2\frac{y}{h}+\frac{y^2}{h^2}\right)+O\left(\frac{a}{h}\right)^3\right\},\end{aligned} \tag{4.18}$$

$$\begin{aligned}\frac{v(x,y)}{\sqrt{(gh)}} &= \left[-\left(\frac{d}{e}\right)^4\left(\frac{y}{d}+\frac{h}{d}\right)f_X-\left(\frac{d}{e}\right)^6\left(\frac{y}{d}+\frac{h}{d}\right)R_X\right.\\ &\quad \left.+\left(\frac{d}{e}\right)^6\frac{1}{6d^3}(y^3+3hy^2-2h^3)f_{XXX}\right]\frac{e}{\sqrt{(dh)}}\\ &= -\left\{\left[\frac{3}{k^2}\left(\frac{a}{h}\right)^3\right]^{\frac{1}{2}}\left(1+\frac{y}{h}\right)cn(\alpha X,k)\,sn(\alpha X,k)\,dn(\alpha X,k)\right\}\\ &\quad \times\left[1-\frac{a}{h}\left(\frac{5k^2+2}{8k^2}\right)-\frac{a}{h}\left(1-\frac{1}{2k^2}\right)\left(2\frac{y}{h}+\frac{y^2}{h^2}\right)\right.\\ &\quad \left.-\frac{1}{2}\frac{a}{h}\left(1-6\frac{y}{h}-3\frac{y^2}{h^2}\right)cn^2(\alpha X,k)\right]+O\left(\frac{a}{h}\right)^{\frac{7}{2}},\end{aligned} \tag{4.19}$$

$$\frac{\Delta p(x,y)}{\rho g h} = \frac{\eta(x)-y}{h} - \frac{d^3 h}{2e^4}\left[\left(\frac{\eta_0}{h}\right)^2 - \left(\frac{y}{h}\right)^2 + 2\frac{\eta_0}{h} - 2\frac{y}{h}\right] U_0 f_{XX}$$

$$= \frac{\eta(x)-y}{h} - \left(\frac{a}{h}\right)^2 \frac{3}{4k^2}\left(2\frac{y}{h}+\frac{y^2}{h^2}\right)$$

$$\times [1-k^2+2(2k^2-1)\,cn^2(\alpha X,k) - 3k^2 cn^4(\alpha X,k)] + O\left(\frac{a}{h}\right)^3. \quad (4.20)$$

These second approximations to the cnoidal wave show that the $(a/h)^2$ terms of Korteweg & de Vries (1895) are not all correct, although their expressions predict nearly the same behaviour as do (4.16) to (4.20). Their results are most nearly correct for the limiting case of the solitary wave ($k = 1$). As a matter of fact, their expressions for the solitary wave profile and its propagation velocity may be written as

$$\left.\begin{aligned} \frac{\eta(x)}{h} &= \frac{a}{h}\operatorname{sech}^2\alpha X - \frac{3}{4}\left(\frac{a}{h}\right)^2 \operatorname{sech}^2\alpha X(1-\operatorname{sech}^2\alpha X) + O\left(\frac{a}{h}\right)^3, \\ \alpha X &= \frac{x}{h}\left(\frac{3}{4}\frac{a}{h}\right)^{\frac{1}{2}}\left(1-\frac{5}{8}\frac{a}{h}\right) + O\left(\frac{a}{h}\right)^{\frac{5}{2}}, \\ \frac{u(\infty)}{\sqrt{(gh)}} &= 1+\frac{1}{2}\frac{a}{h} - \frac{3}{20}\left(\frac{a}{h}\right)^2 + O\left(\frac{a}{h}\right)^3, \end{aligned}\right\} \quad (4.21)$$

and these results are seen to be in exact agreement with the values given by (4.16), (4.17) and (4.18) for this limiting case of $k = 1$.

The solitary wave propagation velocity given by (4.21) is also in exact agreement with the one obtained by Weinstein (1926), after his numerical error is corrected as was done by Long (1956), and also by Hunt (1955) who derived

$$\frac{u(\infty)}{\sqrt{(gh)}} = \left[1+\frac{a}{h} - \frac{1}{20}\left(\frac{a}{h}\right)^2 - \frac{3}{70}\left(\frac{a}{h}\right)^3 + \ldots\right]^{\frac{1}{2}}$$

$$= 1+\frac{1}{2}\frac{a}{h} - \frac{3}{20}\left(\frac{a}{h}\right)^2 + \frac{3}{56}\left(\frac{a}{h}\right)^3 + \ldots. \quad (4.22)$$

Since Weinstein used an entirely different expansion method for his successive approximations to the solitary wave, and his corrected results as given in (4.22) are in agreement with the preceding results derived by Friedrichs's expansion method, therefore we are led to believe that the power series in terms of σ does converge as long as $(a/h) < 1$ and k is sufficiently near unity. Littman (1957) gave an existence proof for cnoidal waves, using a method resembling the one used by Friedrichs & Hyers (1954) to prove the existence of solitary waves, which demonstrated that at least for cnoidal waves Friedrichs's expansion method may only give an asymptotic description.

It is interesting to note that values of k near unity correspond to supercritical flow, or $u > \sqrt{(gh)}$. For example, (4.21) in itself shows that the solitary wave ($k = 1$) can only occur in supercritical flow. However, as first proved by Littman (1957), the cnoidal wave can exist in either subcritical or supercritical flow, but it must be noted that Littman's existence proof is valid only near critical flow, so cnoidal waves may be restricted to near critical flow. The conditions necessary for subcritical flow are best determined from considerations of the average propagation velocity. Following Korteweg & de Vries (1895), we define the average velocity of propagation C, as the constant horizontal velocity which would reduce the resultant horizontal momentum to zero. Therefore, from (3.15),

$$C = \frac{\int_0^\lambda \int_0^{h+\eta} u\,dy\,dx}{\int_0^\lambda \int_0^{h+\eta} dy\,dx}$$

$$= \frac{\sqrt{(gh)}\int_0^\lambda \left[1+\left(\frac{2k^2-1}{2k^2}\right)\frac{a}{h}-\frac{\eta(x)}{h}+O\left(\frac{a}{h}\right)^2\right](h+\eta)\,dx}{h\lambda+\int_0^\lambda \eta(x)\,dx}$$

$$= \sqrt{(gh)}\left[1+\left(\frac{2k^2-1}{2k^2}\right)\frac{a}{h}+O\left(\frac{a}{h}\right)^2\right]\left[1+\frac{1}{\lambda}\int_0^\lambda \frac{\eta(x)}{h}\,dx\right]^{-1}, \tag{4.23}$$

while from (3.11) and (3.12) we obtain

$$\frac{1}{\lambda}\int_0^\lambda \frac{\eta(x)}{h}\,dx = \frac{1}{\lambda}\frac{a}{h}\frac{e}{\alpha}\int_0^{2K(k)} cn^2\left(\frac{\alpha x}{e},k\right)d\left(\frac{\alpha x}{e}\right)$$

$$= \frac{a}{h}\frac{1}{k^2}\left[\frac{E(k)}{K(k)}-(1-k^2)\right]+O\left(\frac{a}{h}\right)^2 = \frac{\overline{\eta}}{h}, \tag{4.24}$$

because (3.12) and figure 1 shows that the wavelength λ is given by

$$\frac{\lambda}{h} = \frac{2K(k)}{h\alpha/e} = 2K(k)\Big/\left[\frac{3}{4k^2}\frac{a/h}{(1+\eta_0/h)^3}\right]^{\frac{1}{2}}$$

$$= \frac{4kK(k)}{(3a/h)^{\frac{1}{2}}}\left[1+\frac{3}{2}\left(\frac{2k^2-1}{k^2}\right)\frac{a}{h}+O\left(\frac{a}{h}\right)^2\right] = \frac{4kK(k)}{(3a/h)^{\frac{1}{2}}}\left[1+O\left(\frac{a}{h}\right)\right]. \tag{4.25}$$

Then, if we substitute (4.24) into (4.23) and retain only the first-order terms, we obtain

$$\frac{C}{\sqrt{(gh)}} = 1+\frac{a}{h}\frac{1}{k^2}\left[\frac{1}{2}-\frac{E(k)}{K(k)}\right]+O\left(\frac{a}{h}\right)^2, \tag{4.26}$$

which is identical to the expression first given by Korteweg & de Vries (1895). It is important to note that the average propagation velocity defined by (4.26) satisfies both definitions proposed by Stokes (1880, p. 203) in this first (or linearized) approximation, as may be seen after integrating (3.15) by noting (4.24)

$$\frac{1}{\lambda}\int_0^\lambda \frac{u(x)}{\sqrt{(gh)}}\,dx = \left[1+\left(\frac{2k^2-1}{2k^2}\right)\frac{a}{h}+O\left(\frac{a}{h}\right)^2-\frac{1}{\lambda}\int_0^\lambda \frac{\eta(x)}{h}\,dx\right] = \frac{C}{\sqrt{(gh)}}+O\left(\frac{a}{h}\right)^2.$$

However, this is only true for the first approximation since the second approximation (4.18) shows that the local velocity now depends upon y also.

The direct effect of the second approximation on the wavelength is shown by solving (4.17) to obtain

$$\frac{\lambda}{h} = \frac{4kK(k)}{(3a/h)^{\frac{1}{2}}}\left[1+\left(\frac{7k^2-2}{8k^2}\right)\frac{a}{h}+O\left(\frac{a}{h}\right)^2\right], \tag{4.27}$$

which conclusively proves that Korteweg & de Vries (1895) were correct in eliminating the higher-order terms from (4.25) and (4.26). A comparison of (4.25) and (4.27) indicates how the second approximation directly alters the coefficients of the terms that must be neglected.

Equation (4.26) shows that the critical velocity ($C = \surd(gh)$) is attained when

$$\frac{E(k)}{K(k)} = \frac{1}{2}, \quad K(k) = 2{\cdot}321, \quad k^2 = 0{\cdot}8261, \quad \frac{\lambda}{h} = \frac{4{\cdot}87}{(a/h)^{\frac{1}{2}}}\left[1+O\left(\frac{a}{h}\right)\right]. \tag{4.28}$$

Consequently, subcritical flow can only occur if $k < 0{\cdot}9$ and $(\lambda/h) < 4{\cdot}87/\surd(a/h)$. Since the generally accepted, experimentally verified, restriction on shallow-water theory is that $\lambda > 10h$, therefore finite amplitude cnoidal waves must have k near unity, and have their average propagation velocities near critical. For example, if $k = 0{\cdot}7$, then $a/h < 0{\cdot}09$ in order to keep $\lambda > 10h$, and this amplitude of wave height is such that it may be better described by the higher-order terms of the small amplitude surface-wave theory of Stokes (1880, pp. 197 and 314). As previously pointed out, the limiting value of $k = 0$ is not a solution of (3.7), and as shown by (3.15) and (4.26) small values of k can make all the velocities negative, so they are physically impossible unless the wave amplitude approaches zero as a limiting value.

5. The limiting height and maximum velocity of cnoidal and solitary waves

A comparison of (3.14) and (4.20) proves that the hydrostatic pressure assumption for finite amplitude shallow-water waves is valid only to the first order of (a/h). Even more important, however, is the fact that (4.19) shows that the vertical velocity variation is no longer monotonic if a/h is sufficiently large. For example, (3.16) gave a linear variation in vertical velocity, whereas (4.19) proves that the increase of v with a/h can be completely reversed for all y near the surface of a wave crest whenever

$$\frac{a}{h} = \frac{8k^2}{9k^2+2} \leqslant \frac{8}{11}, \quad h > a \geqslant |y|, \qquad cn^2(\alpha X, k) \doteqdot \left[1-\frac{3}{4k^2}\frac{a}{h}\left(\frac{x}{h}\right)^2+O\left(\frac{a}{h}\right)^2\right], \tag{5.1}$$

since all $y \doteqdot a$ must be neglected in (4.19) because they form terms of $O(a/h)^3$. This reversal in the variation of the vertical velocity may be considered as defining the limiting height of cnoidal waves, the maximum value occurring for the solitary wave when $k = 1$, so (4.21) and (5.1) yield

$$\left(\frac{a}{h}\right)_{\max} = \frac{8}{11}, \quad \left[\frac{u(\infty)}{\surd(gh)}\right]_{\max} = 1{\cdot}284. \tag{5.2}$$

That is, we have defined the limiting height as the smallest height at which the vertical velocity ceases to be a monotonic function of y. It can now be shown that this provides a rational limiting height and velocity for the solitary wave by noting that for the special case of $k = 1$ we may write (4.13) as

$$\left.\begin{aligned} \frac{\frac{3}{5}}{a+h}\eta_1^2-\eta_1+a &= 0,\\ \eta_1 = \tfrac{5}{6}(a+h)\left[1-\left\{1-\frac{12}{5}\left(\frac{a}{a+h}\right)\right\}^{\frac{1}{2}}\right], \end{aligned}\right\} \tag{5.3}$$

which has a real solution only if

$$\frac{a+h}{a} > \frac{12}{5}, \quad \frac{a}{h} < \frac{5}{7}, \quad \frac{u(\infty)}{\sqrt{(gh)}} < 1{\cdot}281. \tag{5.4}$$

These limiting values are not only in close numerical agreement but are also consistent with the order of approximation involved since $\frac{5}{7} = 0{\cdot}7143$ is based on the equation for η which is of the order $(a/h)^2$, whereas $\frac{8}{11} = 0{\cdot}7273$ is based on the equation for v which is of the order $(a/h)^{\frac{5}{2}}$. These values are in excellent agreement with recent experimental investigations by Daily & Stephan (1952), and Ippen & Kulin (1955), who have shown that all steady-state solitary waves have $(a/h) < 0{\cdot}72$ rather than the limiting value of $0{\cdot}782$ as originally evaluated by McCowan (1894), or the value of $0{\cdot}827$ as recently obtained by Yamada (1957).

It must be noted that (5.2) is the first theoretical evaluation of the limiting height of solitary waves that is based upon the vertical velocity variation. The previous limiting heights have been primarily based upon approximate numerical calculations of a profile having a sharp peak with an enclosed angle of 120°, in accordance with Stokes's (1880, p. 227) conjecture that this would define the limiting height since it would correspond to a relative local velocity of zero at the crest. However, Korteweg & de Vries (1895) have proved that any finite amplitude, shallow-water profile that did not correspond to (4.21) would not be steady with respect to time; consequently, the sharp peak 120° wave crest should always be higher than the limiting value of $\frac{8}{11}$ since it corresponds to an unstable wave that has exceeded the breaking height. Experimental observations by the author have shown that when the limiting height of $\frac{8}{11}$ is approached, the wave crest breaks unsymmetrically with a round crest instead of a sharp peak.

For cnoidal waves, (5.1) would define the limiting height for the vertical velocity reversal, and (4.18) would give the corresponding horizontal velocity component. However, for a cnoidal wave it is generally more desirable to refer the wave profile to the average depth $(h+\overline{\eta})$ (see figure 1) rather than the depth h that occurs beneath the wave tough. Only in the limiting case of $k = 1$ does h represent the still-water depth at an infinite distance from the solitary wave. For values of $k < 1$ the average depth $(h+\overline{\eta})$ would correspond to the still-water depth since it represents the height of the water surface if the cnoidal waves were flattened. The first approximation for $\overline{\eta}$ is given by (4.24), and as would be expected, $\overline{\eta} = 0$ for the solitary wave ($k = 1$, $\lambda \sim K \to \infty$). It is interesting to note from (3.15) and (4.26) that $(u-C)$, the relative velocity of the water particles, is given by

$$\frac{u-C}{\sqrt{(gh)}} = \frac{\overline{\eta}}{h} - \frac{\eta(x)}{h}. \tag{5.5}$$

Consequently, the relative velocity of the water particles is zero at the average depth $(h+\bar{\eta})$, and is in the direction of the wave propagation only when $\eta < \bar{\eta}$.

The above results are for the first approximation to cnoidal waves, and now may be carried out to the second approximation by introducing (4.16) into (4.24) to obtain

$$\frac{\bar{\eta}}{h} = \frac{1}{\lambda}\int_0^\lambda \frac{\eta(x)}{h}\,dx = \left\{\left[\frac{a}{h} - \frac{3}{4}\left(\frac{a}{h}\right)^2\right]\frac{1}{k^2}\left[\frac{E(k)}{K(k)} - (1-k^2)\right]\right.$$
$$\left. + \left(\frac{a}{h}\right)^2 \frac{1}{3k^4}\left[(4k^2-2)\frac{E(k)}{K(k)} + (2-5k^2+3k^4)\right] + O\left(\frac{a}{h}\right)^3\right\}$$
$$= \left\{\frac{a}{h}\frac{1}{k^2}\left[\frac{E(k)}{K(k)} - (1-k^2)\right] + \left(\frac{a}{h}\right)^2 \frac{1}{12k^4}\left[-(8-7k^2)\frac{E(k)}{K(k)} + (1-k^2)(8-3k^2)\right]\right\}. \tag{5.6}$$

As before, $\bar{\eta} = 0$ for the limiting case of the solitary wave ($k = 1$, $\lambda \sim K \to \infty$). In addition, through a rather fortunate circumstance, the second approximation for $\bar{\eta}$ gives approximately the same value as does the first approximation (4.24) because a numerical comparison shows that

$$\frac{E(k)}{K(k)} \doteqdot \frac{(1-k^2)(8-3k^2)}{(8-7k^2)} = 1 - \tfrac{1}{2}k^2 - \tfrac{1}{16}k^4 - \tfrac{7}{128}k^6 + \dots. \tag{5.7}$$

This numerical approximation seems to be remarkably accurate for small values of k. Of course, as previously pointed out, the finite-amplitude cnoidal wave theory is only valid for k approaching unity, so (5.2) may be considered as defining the limiting values of the heights and propagation velocities of cnoidal waves, with $\bar{\eta}$ closely approximated by (4.24).

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